EPFL

Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 7

Exercise 1:

(a) *Sufficient conditions for properness*: Let X and Y be topological spaces and let $F\colon X \to Y$ be a continuous map. Prove the following assertions:

(i) If X is compact and Y is Hausdorff, then F is proper.

(ii) If F is a topological embedding with closed image, then F is proper.

(iii) If Y is Hausdorff and F has a continuous *left inverse*, i.e., a continuous map $G\colon Y \to X$ such that $G \circ F = \mathrm{Id}_X$, then F is proper.

(b) Let M be a smooth manifold and let S be an embedded submanifold of M. Show that S is properly embedded if and only if S is a closed subset of M.

(c) *Global graphs are properly embedded*: Let $f\colon M \to N$ be a smooth map between smooth manifolds. Show that the graph $\Gamma(f)$ of f is a properly embedded submanifold of $M \times N$.

Exercise 2:
Fix $n \geq 0$. Using

(i) the local slice criterion, and

(ii) the regular level set theorem,

show that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$.

Exercise 3:
Consider the smooth curve

$$\beta\colon (-\pi, \pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t, \sin t)$$

from *Example 4.5*(2). Show that its image is not an embedded submanifold of $\mathbb{R}^2$.

[Be careful: this is not the same as showing that β is not a smooth embedding.]

Exercise 4 (to be submitted by Thursday, 06.11.2025, 16:00):

(a) Consider the map

$$F\colon \mathbb{R}^4 \to \mathbb{R}^2,\ (x, y, s, t) \mapsto (x^2 + y,\ x^2 + y^2 + s^2 + t^2 + y).$$

Show that $(0, 1) \in \mathbb{R}^2$ is a regular value of F, and that the level set $F^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

(b) Consider the smooth function

$$\Phi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^2.$$

Given $c \in \mathbb{R}$, examine whether the corresponding level set $\Phi^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.

(c) Determine the regular values of the smooth function

$$\Psi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^3$$

and draw the level sets corresponding to its critical values.

Exercise 5:

(a) *Restricting the domain of a smooth map*: If $F\colon M \to N$ is a smooth map and if $S \subseteq M$ is an immersed or embedded submanifold, then the restriction $F|_S\colon S \to N$ is smooth.

(b) *Restricting the codomain of a smooth map*: Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold, and let $G\colon N \to M$ be a smooth map whose image is contained in S. If G is a continuous map from N to S, then $G\colon N \to S$ is smooth.

(c) Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Then every smooth map $G\colon N \to M$ whose image is contained in S is also smooth as a map from N to S.